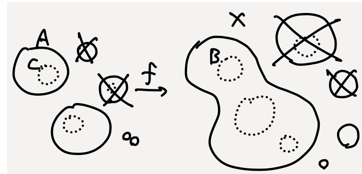
Chapter 11

The homotopy excision and suspension theorems

1. Statement of the homotopy excision theorem

Def. 1 $n \geq 1$

Cと交わらないAのp-con成分は
Bと交わらないXのp-con成分に行く.



(空間対の)
$f : (A, C) \longrightarrow (X, B)$ が n-equivalence

$\overset{\text{def}}{\Longleftrightarrow}$

- $\mathrm{Im}\begin{pmatrix}\pi_0(A)\\ \uparrow \\ \pi_0(C)\end{pmatrix} = f_*^{-1}\,\mathrm{Im}\begin{pmatrix}\pi_0(X)\\ \uparrow \\ \pi_0(B)\end{pmatrix}$

  Cと交わるAのp-con成分　　Bと交わるXのp-con成分

- 基点を(Cに)どう取っても、
  $f_* : \pi_q(A, C) \longrightarrow \pi_q(X, B)$ は $(1 \overset{?}{\leq})\, q < n$ で bij.
  $q = n$ で surj.

Rmk.

$$\cdots \to \pi_n(A, C, *) \to \cdots \to \pi_q(A, C, *) \to \cdots \to \pi_0(C) \to \pi_0(A) \quad : \text{ex.}$$
$$\downarrow \qquad\qquad \wr\!\downarrow \qquad\qquad \downarrow \qquad \downarrow$$
$$\cdots \to \pi_n(B, X, \star) \to \cdots \to \pi_q(X, B, \star) \to \cdots \to \pi_0(B) \to \pi_0(X) \quad : \text{ex.}$$

$$\begin{array}{ccccc} 1 & \to & \mathrm{Im}\begin{pmatrix}\pi_0(A)\\ \uparrow\\ \pi_0(C)\end{pmatrix} & \to & \pi_0(A) \quad : \text{ex.} \\ & & \downarrow & \text{p.b.} & \downarrow f_* \\ 1 & \to & \mathrm{Im}\begin{pmatrix}\pi_0(X)\\ \uparrow\\ \pi_0(B)\end{pmatrix} & \to & \pi_0(X) \quad : \text{ex.} \end{array}$$

―― P69の n-equiv. との互換性 ――

$\left(A \xrightarrow{f} X \text{ が } n\text{-equiv.} \Leftrightarrow \forall * \in A,\ \pi_q(A) \to \pi_q(X) \text{ は } \begin{array}{l} q < n \text{ で bij} \\ q = n \text{ で surj.} \end{array} \right)$

☆ $A \xrightarrow{f} X$ が n-equiv. $\Leftrightarrow$ $\forall * \in X$, $(A, \{*\}) \to (X, \{f(*)\})$ が n-equiv.
かつ $\pi_0(A) \to \pi_0(X)$ が全射.

$\left(\because\right.$ $\Rightarrow$: $\mathrm{Im}\begin{pmatrix}\pi_0(A)\\ \uparrow\\ \pi_0(*)\end{pmatrix} = \{(*\text{を含む}A\text{のp-con成分})\}$

$\mathrm{Im}\begin{pmatrix}\pi_0(X)\\ \uparrow\\ \pi_0(f(*))\end{pmatrix} = \{(f(*)\text{を含む}X\text{のp-con成分})\}$

仮定より $\pi_0(A) \to \pi_0(X)$: bij. なので OK.

$\Leftarrow$: $\pi_0(A) \to \pi_0(X)$ の単射のみ示せばよいが、

もし $A \supseteq A_1, A_2 \xrightarrow{f_*} \otimes \in X$ なら $x \in A_1$ を基点にしたとき

$f_*\,\mathrm{Im}\begin{pmatrix}\pi_0(X)\\ \uparrow\\ \pi_0(f(x))\end{pmatrix}$ は2点からなり $\neq \mathrm{Im}\begin{pmatrix}\pi_0(A)\\ \uparrow\\ \pi_0(*)\end{pmatrix}$.

Def. 2　$n \geq 0$

(A, C) が n-connected

$$\overset{\text{def}}{\iff} \left[\begin{array}{l} \circ\ \pi_0(C) \longrightarrow \pi_0(A) \text{ が surj} \\ \circ\ \text{基点を}(C\text{に})\text{どう取っても、} \\ \quad \pi_q(A, C) = 0 \quad \text{for } 1 \leq q \leq n \end{array} \right.$$

$\iff$ $C \hookrightarrow A$ が n-equivalence

$\iff$ $\forall * \in C,$ $(C, \{*\}) \hookrightarrow (A, \{*\})$ が n-equiv. かつ $\pi_0(C) \twoheadrightarrow \pi_0(A)$.

$$\iff \forall * \in C, \quad \begin{array}{lcll} \to \pi_n(C) & \twoheadrightarrow & \pi_n(A) & \to \pi_n(A, C) = 0 \\ \to \pi_{n-1}(C) & = & \pi_{n-1}(A) & \to \pi_{n-1}(A, C) = 0 \\ \quad \vdots & & \quad \vdots & \quad \vdots \\ \to \pi_1(C) & = & \pi_1(A) & \to \pi_1(A, C) = 0 \\ \to \pi_0(C) & & \pi_0(A) & \end{array}$$

○ $(A, C) \longrightarrow (X, B)$ が n-equiv.

$\Leftrightarrow$ $^\forall$基点, $\quad \pi_n(A, C) \twoheadrightarrow \pi_n(X, B)$

$\pi_i(A, C) = \pi_i(X, B) \quad (1 \leq i < n)$

$\pi_0(A, C) \hookrightarrow \pi_0(X, B)$

○ $A \longrightarrow X$ が n-equiv.

$\Leftrightarrow$ $^\forall$基点, $\quad \pi_n(A, *) \twoheadrightarrow \pi_n(X, *)$

$\pi_i(A, *) = \pi_i(X, *) \quad (1 \leq i < n)$

$\pi_0(A, *) = \pi_0(X, *)$

○ (X, A) が n-conn. $\Leftrightarrow$ $A \hookrightarrow X$ が n-equiv.

○ X が n-conn. $\Leftrightarrow$ $^\forall$基点, $\pi_i(X) = 0 \quad (0 \leq i \leq n)$

$\Leftrightarrow$ $^\forall$基点, $(X, *)$ が n-conn.

Thm.3 (Homotopy excision)

- $(X; A, B)$: 切除対で $C \equiv A\cap B \neq \emptyset$.
- (A, C) : $(m-1)$-conn. $(m \geq 2)$
- (B, C) : $(n-1)$-conn. $(n \geq 1)$

$\Rightarrow$ $(A, C) \hookrightarrow (X, B)$ は $(m+n-2)$-equiv.

$$
\begin{tikzcd}
 & X & {= A^\circ \cup B^\circ} \\
A \arrow[ur, dash, "\text{exc.}"] & {} \arrow[u, phantom, "\nearrow\!\!\!\nearrow_{m+n}"] & B \arrow[ul, dash] \\
 & C \arrow[ul, dash, "m"] \arrow[ur, dash, "n"'] \arrow[u, phantom, "\cap"] & {= A\cap B}
\end{tikzcd}
$$

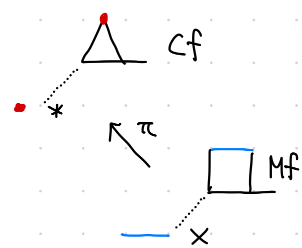
証明はあとまわしにして、これを使ってわかることを述べる。

Thm.4 $n \geq 2$.

- $f : X \longrightarrow Y$: $(n-2)$-conn. の間の $(n-1)$-equiv.

基点との組として

$\Rightarrow \left[\begin{array}{l} \cdot\ \pi : (Mf, X) \longrightarrow (Cf, *) \text{ は } (2n-2)\text{-equiv.} \\ \cdot\ Cf \text{ は } (n-1)\text{-conn.} \end{array}\right.$



- $f : X \longrightarrow Y$: $(n-1)$-conn. (の間の $(n-1)$-equiv.)

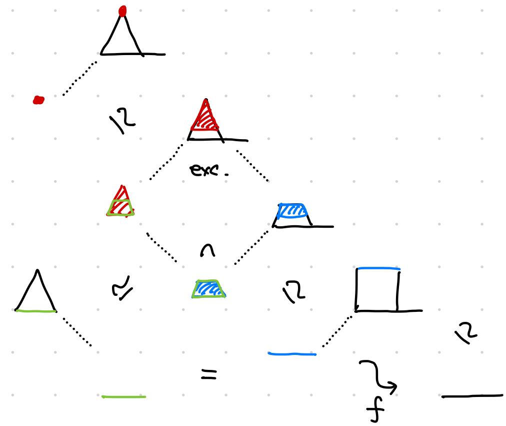
$\Rightarrow \left[\ \cdot\ \pi : (Mf, X) \longrightarrow (Cf, *) \text{ は } (2n-1)\text{-equiv.}\right.$

注意: X, Y : $(n-2)$-conn. なら自動的に $f : X \to Y$ は $(n-2)$-equiv.

prf.

$\boxed{0}$

$$\begin{tikzcd}[column sep=small]
 & & & Cf \arrow[dl, dash] \arrow[dr, equal] & & & & \\
 & & \{*\} \arrow[r, phantom, "\simeq"] \arrow[dr, phantom, "\simeq"] & & Cf \arrow[dr, dash] \arrow[dl, dash] & & & \\
 & & & B \arrow[dl, phantom, "\simeq"] \arrow[dr, dash, "n-1"] \arrow[rr, phantom, "\overset{\mathrm{exc.}}{\nwarrow 2n-2}"] & & A \arrow[uull, bend right, "\pi"', "2n-2"] \arrow[dr, phantom, "\simeq"] & & \\
 & & CX \arrow[dr, dash, "n-1"'] \arrow[r, phantom, "\simeq"] & & C \arrow[ur, dash, "n-1"] \arrow[dl, phantom, "\simeq"] \arrow[dr, phantom, "\simeq"] & & Mf \arrow[dl, dash, "n-1"] \arrow[dr, phantom, "\simeq"] & \\
 & & \boxed{2} & \underset{n-2}{X} \arrow[rr, equal] & & \underset{n-2}{X} \arrow[rr, "n-1", "f"'] \arrow[ur, phantom, "\nwarrow \boxed{1}"] & & \underset{n-2}{Y}
\end{tikzcd}$$



$\boxed{1}$

$$\begin{array}{ccccccc}
\cdots \to & \pi_{n-1}(X) & \overset{①}{\twoheadrightarrow} & \pi_{n-1}(Mf) & \xrightarrow{\overset{②}{\text{零}}} & \pi_{n-1}(Mf, X) & \overset{②}{\text{inj.}} \quad f,2 \quad = 0 \quad ③ \\
\overset{②}{\xrightarrow{\partial}} & \overset{①}{\pi_{n-2}(X)}_{=0} & \xrightarrow{\sim} & \overset{①}{\pi_{n-2}(Mf)}_{=0} & \to & \pi_{n-2}(Mf, X) & = 0 \\
\xrightarrow{\partial} & \cdots_{=0} & \xrightarrow{\sim} & \cdots_{=0} & & \vdots & = 0 \\
\xrightarrow{\partial} & \pi_0(X)_{=0} & \xrightarrow{\sim} & \pi_0(Mf)_{=0} & & &
\end{array}$$

$\because X : (n-2)$-conn. $\quad \because f : (n-1)$-equiv. $\quad \because Y : (n-2)$-conn.

i.e. $(Mf, X) : (n-1)$-conn.

$\boxed{2}$

$$\begin{array}{ccccccc}
 & & & & & \pi_n(CX, X) & =0 \quad ③ \\
\overset{②}{\xrightarrow{\sim}} & \pi_{n-1}(X)_{=0} & \twoheadrightarrow & \overset{①}{\pi_{n-1}(CX)}_{=0} & \to & \pi_{n-1}(CX, X) & f,2 \quad =0 \\
\xrightarrow{\sim} & \overset{①}{\pi_{n-2}(X)}_{=0} & \twoheadrightarrow & \pi_{n-2}(CX)_{=0} & \to & \pi_{n-2}(CX, X) & =0 \\
\xrightarrow{\sim} & \cdots & \twoheadrightarrow & \cdots & & & \vdots \\
\xrightarrow{\sim} & \pi_0(X)_{=0} & \overset{②}{\twoheadrightarrow} & \pi_0(CX)_{=0} & & &
\end{array}$$

$\because X : (n-2)$-conn. $\quad \because CX \simeq *$

i.e. $(CX, X) : (n-1)$-conn. (n)

$\square$

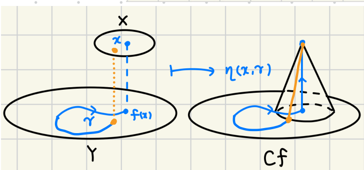
Cor.5 $n \geq 2$

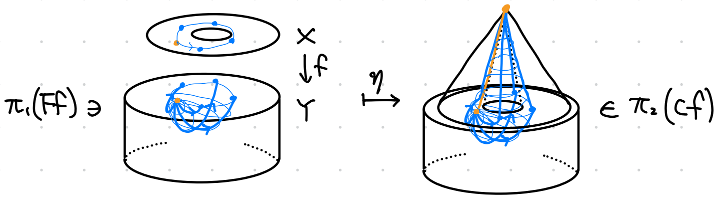
$f : X \to Y$: well-pointed な $(n-1)$-conn. の間の based map

$\Rightarrow$
- Cf は $(n-1)$-conn.
- $\pi_n(Mf, X) \longrightarrow \pi_n(Cf, *)$ は iso.
- $\eta : Ff \to \Omega Cf$ による
  $\pi_{n-1}(Ff) \to \pi_n(Cf)$ は iso.　（Cf, Mf は reduced の方）

Rmk.

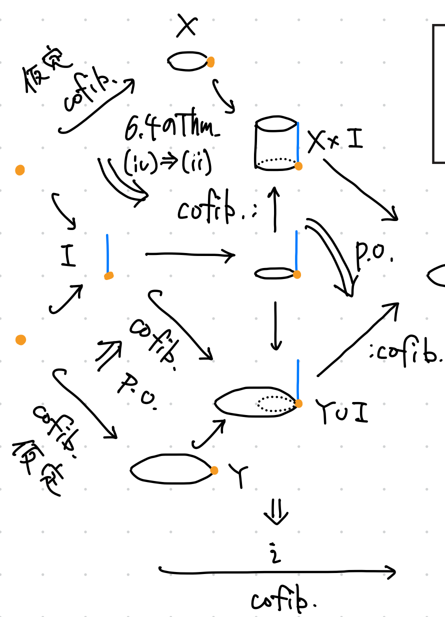
$f : X \to Y$ に対し、$\eta : Ff \longrightarrow \Omega Cf$, $\varepsilon : \Sigma Ff \longrightarrow \underset{CX \cup_f Y}{\overset{\shortparallel}{Cf}}$ を $\underset{\substack{\in \\ X}}{\eta(x}, \overset{\gamma(1)=f(x)}{\underset{\substack{\in \\ PY}}{\gamma}})(t) = \varepsilon(x, \gamma, t) = \begin{cases} \gamma(2t)^{\in Y} & 0 \leq t \leq \frac{1}{2} \\ (x, 2t-1)_{\in CX} & \frac{1}{2} \leq t \leq 1 \end{cases}$ と定める。



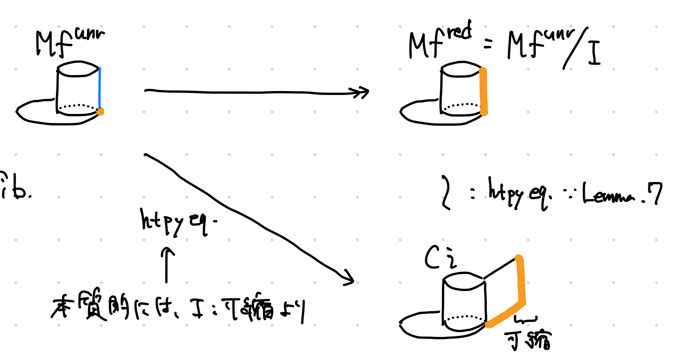


prf.

- well-pointed なので $Mf^{unr} \twoheadrightarrow Mf^{red}$, $Cf^{unr} \twoheadrightarrow Cf^{red}$ は htpy-equiv.



THEOREM. *Let A be a closed subspace of X. Then the following are equivalent:*
(i) (X, A) *is an NDR-pair.*
(ii) $(X \times I, X \times \{0\} \cup A \times I)$ *is a DR-pair.*
(iii) $X \times \{0\} \cup A \times I$ *is a retract of* $X \times I$.
(iv) *The inclusion* $i : A \longrightarrow X$ *is a cofibration.*



Cf も同様.

○ 前Thm.より Cf^{unr} : $(n-1)$-conn. なので Cf^{red} : $(n-1)$-conn.

π : $(2n-1)$-equiv. なので $n \geq 2$ より $\pi_n(Mf, X) \xrightarrow{\sim} \pi_n(Cf, *)$

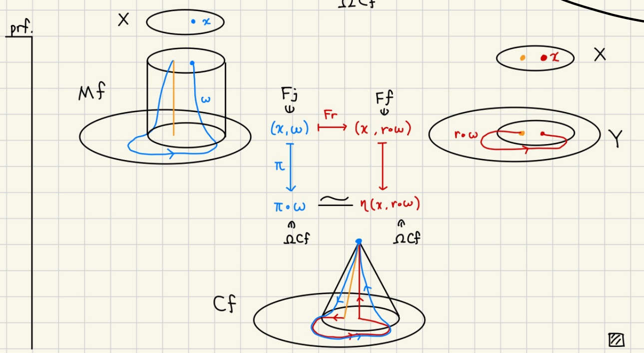
○

<u>Lem. 25</u> $f: X \to Y$ に対し, $j: X \hookrightarrow Mf$, $r: Mf \twoheadrightarrow Y$: retraction, $\pi: Fj \to \Omega Cf$ を商写像 $Mf \to Cf$ から誘導される写像とするとき, 下の図式はホモトピー可換.

$$\begin{tikzcd}
Fj = X \times_j PMf \arrow[rr, "Fr = \mathrm{id} \times Pr\ (r\circ -)"] \arrow[dr, "\pi"'] & & X \times_f PY = Ff \arrow[dl, "\eta"] \\
& \Omega Cf &
\end{tikzcd}$$

$$\begin{tikzcd}
\pi_n(Mf, X) \arrow[d, phantom, "\shortparallel"] & & \\
\pi_{n-1}(P(Mf; *, X)) \arrow[d, phantom, "\shortparallel"] & & \text{ここは?} \\
\pi_{n-1}(Fj) \arrow[rr] \arrow[dr, "\wr"'] & \circlearrowleft & \pi_{n-1}(Ff) \arrow[dl] \\
& \pi_{n-1}(\Omega Cf) \arrow[d, phantom, "\shortparallel"] & \text{ここを知りたい} \\
& \pi_n(Cf) &
\end{tikzcd}$$

prf.



$$\begin{tikzcd}
Ff \arrow[r, "\text{fib.}"] & X \arrow[r, "f"] \arrow[d, phantom, "\shortparallel"] & Y \arrow[d, phantom, "\wr"] \\
& X \arrow[r, "\text{cofib.}"] & Mf
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
\arrow[r] & \pi_n(Ff) \arrow[r] & \pi_n(X) \arrow[r] \arrow[d, phantom, "\shortparallel"] & \pi_n(Y) \arrow[r] \arrow[d, phantom, "\shortparallel"] & \pi_{n-1}(Ff) \arrow[r] \arrow[d, phantom, "\shortparallel"] & \pi_{n-1}(X) \arrow[r] \arrow[d, phantom, "\shortparallel"] & {} \\
& \arrow[r] & \pi_n(X) \arrow[r] & \pi_n(Mf) \arrow[r] & \pi_n(Mf, X) \arrow[r] & \pi_{n-1}(X) \arrow[r] & {}
\end{tikzcd}$$

$\square$.

Thm. 6

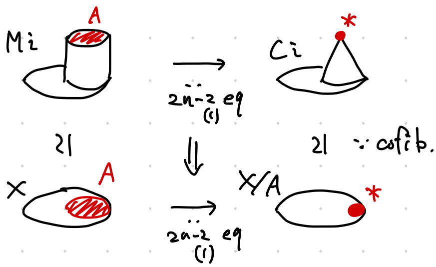
$i : \underset{n-2 \text{ conn.}}{A} \xrightarrow[\text{eq.}]{n-1} \underset{n-2 \text{ conn.}}{X}$: cofibration.

$\Rightarrow (X, A) \longrightarrow (X/A, *)$ は $(2n-2)$-equiv.

$i : \underset{n-1 \text{ conn.}}{A} \xrightarrow[\text{eq.}]{n-1} \underset{n-1 \text{ conn.}}{X}$: cofibration.

$\Rightarrow (X, A) \longrightarrow (X/A, *)$ は $(2n-1)$-equiv.

prf.



□.

2. The Freudenthal suspension theorem

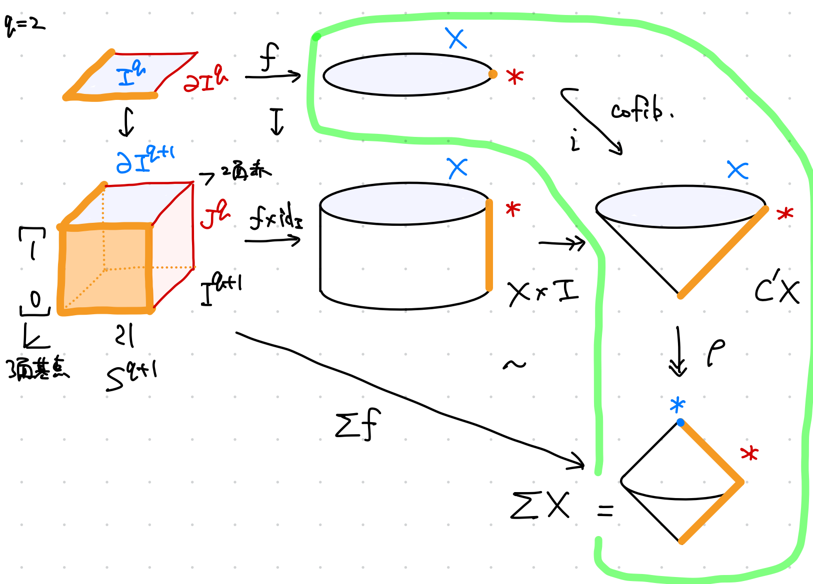
Thm. 7 (Freudenthal suspension). $n \geq 1$

X : well pointed, $(n-1)$-conn.

$\Rightarrow \quad \Sigma : \pi_q(X) \longrightarrow \pi_{q+1}(\Sigma X)$ は $q < 2n-1$ で bij. $q = 2n-1$ で surj.

$$\begin{array}{ccc} \in & & \in \\ S^q & & S^{q+1} = S^q \wedge S^1 \\ f \downarrow & \overset{\Sigma}{\mapsto} & f \downarrow \quad \downarrow id \\ X & & X \wedge S^1 = \Sigma X \end{array}$$

prf. 逆転した錐 $C'X$ = 



$$\begin{tikzcd}
 & \pi_q(C'X) = 1 & \\
\pi_q(X) \arrow[ur] \arrow[ddr, "\Sigma"'] & \text{ex.} & 1 = \pi_{q+1}(C'X) \arrow[dl] \\
 & \pi_{q+1}(C'X, X, *) \arrow[ul, "\partial", "\cong"'] \arrow[d, "p_*"] & \\
 & \pi_{q+1}(\Sigma X) &
\end{tikzcd}$$

X : $(n-1)$-conn.
CX : 可縮 より $i : \underset{n-1\ \text{conn.}}{X} \xrightarrow[\text{eq}]{n} \underset{n-1\ \text{conn.}}{C'X}$: cofibration から p : $2n$-eq.

□

Thm. 8

- $\pi_n(S^n) = \mathbb{Z}$ $(n \geq 1)$
- $\Sigma : \pi_n(S^n) \longrightarrow \pi_{n+1}(S^{n+1})$ は iso.

prf. $\pi_2(S^2) = \mathbb{Z}$ だった。S^2 : 1-conn. で $2 < 2 \times 2 - 1$ より $n \geq 2$ での主張は OK.

$n=1$ は Hopf bundle の連結射を追うと分かるらしい 💦

3. Proof of the homotopy excision theorem

Thm. 3 (Homotopy excision)

- $(X; A, B)$: tp除対で $C = A \cap B \neq \emptyset$.
- (A, C) : $(m-1)$-conn. $(m \geq 2)$
- (B, C) : $(n-1)$-conn. $(n \geq 1)$

$\Rightarrow$ $(A, C) \hookrightarrow (X, B)$ は $(m+n-2)$-equiv.

$$\begin{tikzcd}
 & X \arrow[r, phantom, "= A^\circ \cup B^\circ"] & \\
A \arrow[ur, dash] \arrow[ur, phantom, "\text{exc.}"] & \arrow[ul, phantom, "\nearrow\!\!\!\nearrow_{m+n}"] & B \arrow[ul, dash] \\
 & C \arrow[ul, dash, "m"] \arrow[ur, dash, "n"'] \arrow[r, phantom, "= A \cap B"] &
\end{tikzcd}$$

Prop. 9. $X \supseteq A \supseteq B$

$$\begin{tikzcd}
\pi_q(B) \arrow[r] \arrow[d, equal] & \pi_q(A) \arrow[r] \arrow[d] & \pi_q(A,B) \arrow[r] \arrow[d] & \pi_{q-1}(B) \arrow[r] \arrow[d, equal] & \pi_{q-1}(A) \arrow[r] \arrow[d] & \pi_{q-1}(A,B) \arrow[r] \arrow[d] & {} \\
\pi_q(B) \arrow[r] \arrow[d] & \pi_q(X) \arrow[r] \arrow[d, equal] & \pi_q(X,B) \arrow[r] \arrow[d] & \pi_{q-1}(B) \arrow[r] \arrow[d] & \pi_{q-1}(X) \arrow[r] \arrow[d, equal] & \pi_{q-1}(X,B) \arrow[r] \arrow[d] & {} \\
\pi_q(A) \arrow[r] & \pi_q(X) \arrow[r] & \pi_q(X,A) \arrow[r] & \pi_{q-1}(A) \arrow[r] & \pi_{q-1}(X) \arrow[r] & \pi_{q-1}(X,A) \arrow[r] & {}
\end{tikzcd}$$

: ex.

Def. 10

$$\pi_q(X; A, B) := \pi_{q-1}\big(P(X; *, B),\ P(A; *, C)\big)$$

$$= \left[\left(I^q; I^{q-2} \times \{1\} \times I,\ I^{q-1} \times \{1\},\ J^{q-2} \times I \cup I^{q-1} \times \{0\}\right), (X; A, B, *)\right]$$

Prop. 11 $P(X; *, B) \supseteq P(A; *, C) \supseteq \{*\}$ により

$$\cdots \to \pi_{q+1}(A, C) \to \pi_{q+1}(X, B) \to \pi_{q+1}(X; A, B) \to \pi_q(A, C) \to \pi_q(X, B) \to \pi_q(X; A, B) \to \cdots$$

: ex.

Thm. 3'

$2 \leq q \leq m+n-2$ で $\pi_q(X; A, B) = 0$

outline (May)

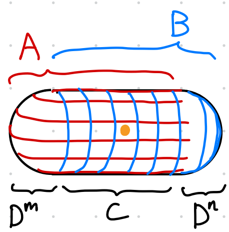
① +CW approximation+ $\quad A \overset{\text{dim rel.cell}\,\geq m}{\text{—}} C \overset{\text{dim rel.cell}\,\geq n}{\text{—}} B$ としてよい.

② $A \text{ — } C$ relative CWとしての cell がひとつの場合に帰着

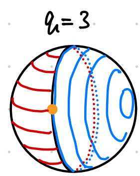
非自明パート: five lemma

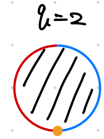
③ $B \text{ — } C$ relative CWとしての cell がひとつの場合に帰着

④



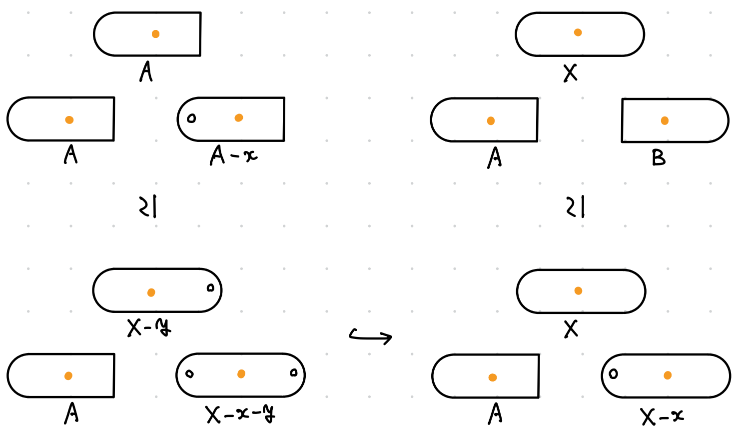
$X \xleftarrow{f}$ $\qquad \pi_q(X;A,B) \ni f$





$2\leq q\leq n+m-2$ で

こういう f が null-homotopic であることを示す.



$$
\begin{array}{ccc}
0 \overset{!}{=} \pi_q(A;A,A-x) & & \pi_q(X;A,B) \\
\| & & \| \\
\pi_q(X-y;A,X-x-y) & \longrightarrow & \pi_q(X;A,X-x)
\end{array}
$$

$\pi_q(X;A,B)$: 知りたい.

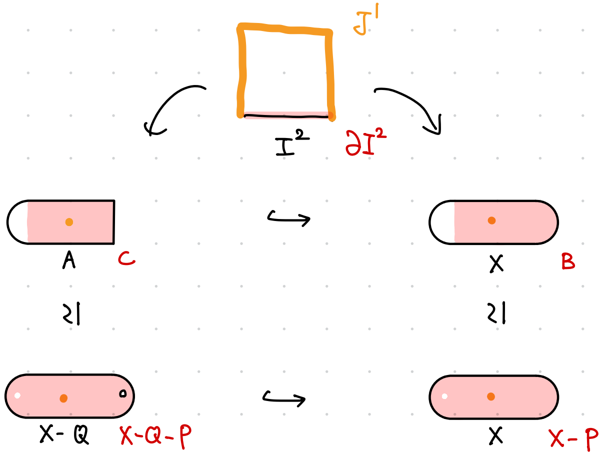
$\pi_q(X-y;A,X-x-y) \longrightarrow \pi_q(X;A,X-x)$: surj. ならよい.

outline (Hatcher) CW複体に限っている.

①

$$\begin{tikzcd} A \arrow[dr, dash, "\text{m-cell のみ}"'] & & B \arrow[dl, dash, "\text{n-cell のみ}"] \\ & C & \end{tikzcd}$$

の場合



$$\begin{tikzcd} \pi_q(A, C) \arrow[r, "\text{知りたい}"] \arrow[d, equal] & \pi_q(X, B) \arrow[d, equal] \\ \pi_q(X-Q, X-Q-P) \arrow[r, "\text{調べる}"] & \pi_q(X, X-P) \end{tikzcd}$$

$\begin{cases} q < m+n-2 \text{ で bij ?} \\ q = m+n-2 \text{ で surj ?} \end{cases}$

〈Key Construction〉 非自明パート : piecewise linear 近似

②

$$\begin{tikzcd} & B \arrow[dl, dash, "\text{(}\geq n\text{)-cell}"] \\ C & \end{tikzcd}$$

の場合

③

$$\begin{tikzcd} A \arrow[dr, dash, "\text{(}\geq m\text{)-cell}"'] & \\ & C \end{tikzcd}$$

(i.e. 一般) の場合

非自明パート : five lemma

†Piecewise Linear approximation†

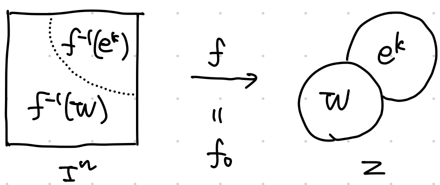
To fill in the missing step in this argument we will need a technical lemma about deforming maps to create some linearity. Define a **polyhedron** in $\mathbb{R}^n$ to be a subspace that is the union of finitely many convex polyhedra, each of which is a compact set obtained by intersecting finitely many half-spaces defined by linear inequalities of the form $\sum_i a_i x_i \le b$. By a **PL (piecewise linear) map** from a polyhedron to $\mathbb{R}^k$ we shall mean a map which is linear when restricted to each convex polyhedron in some such decomposition of the polyhedron into convex polyhedra.

Lemma 4.10. *Let $f : I^n \to Z$ be a map, where Z is obtained from a subspace W by attaching a cell e^k. Then there is a homotopy $f_t : (I^n, f^{-1}(e^k)) \to (Z, e^k)$ rel $f^{-1}(W)$ from $f = f_0$ to a map f_1 for which there is a polyhedron $K \subset I^n$ such that:*

(a) *$f_1(K) \subset e^k$ and $f_1 | K$ is PL with respect to some identification of e^k with $\mathbb{R}^k$.*

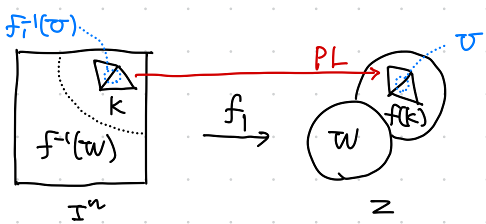
(b) *$K \supset f_1^{-1}(U)$ for some nonempty open set U in e^k.*

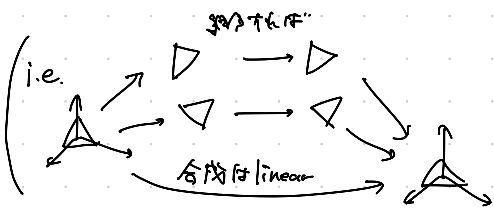
つまり.



が与えられたとき.

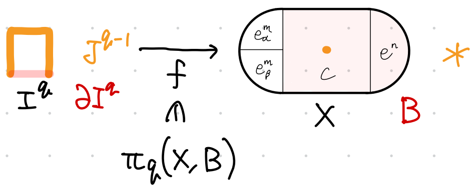
$f^{-1}(W)$ を保つ $\Downarrow$ 対のホモトピーがあって





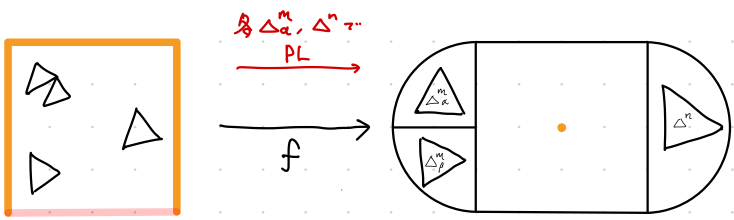
prf.

① m-cell のみ $\{e_\alpha^m\}$ — A — C — B — n-cell ひとつ e^n の場合.



$*$ を取る. 像はcptなので $\{e_\alpha^m\}$, e_n のうち有限個としか交わらない.

f に +PL近似+ を有限回使い,



におきかえてよい.
さらにΔたちについて
全射になるようにしてよい.

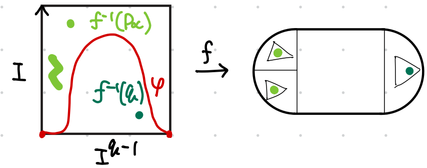
Claim. $q \le m+n-2$ なら、$P=\{p_\alpha \in \Delta^m_\alpha\}$, $q\in\Delta^n$ と $\varphi: I^{q-1}\to[0,1)$ があり

(a) $f^{-1}(q)$ は φ のグラフの下

(b) $\forall\alpha$, $f^{-1}(p_\alpha)$ は φ のグラフの上

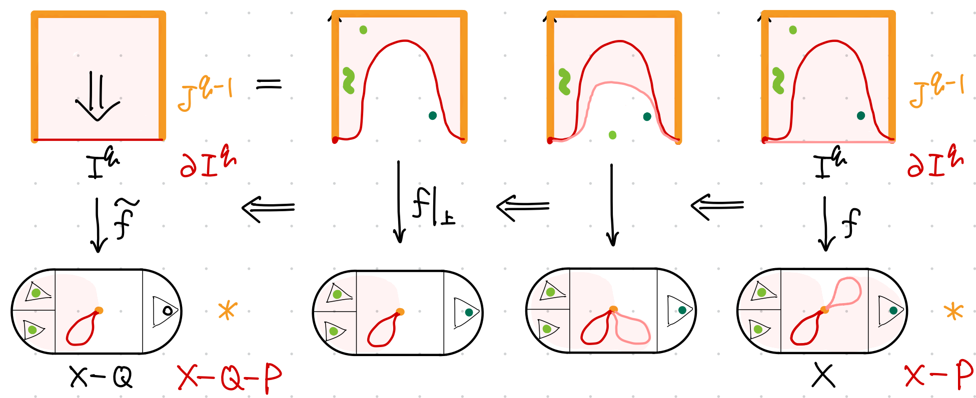
(c) ∂I^{q-1} 上で $\varphi\equiv 0$

$I^q = I^{q-1}\times I$ 内



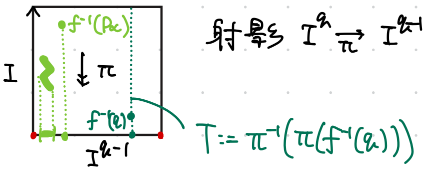
これができれば、$q\le m+n-2$ で

$$\begin{tikzcd}
\pi_q(A,C) \arrow[rr] \arrow[d, equal] & & \pi_q(X,B) \arrow[d, equal] \\
\pi_q(X-Q, X-Q-P) \arrow[rr] & & \pi_q(X, X-P) \\
\cup & & \cup
\end{tikzcd}$$



が構成できるので、$\pi_q(A,C) \to \pi_q(X,B)$: 全射.

Claim. の prf.　　$q \le m+n-2$



射影 $I^q \xrightarrow{\pi} I^{q-1}$ による $f^{-1}(P_\alpha)$ たちの像と

$f^{-1}(q)$ の像が disjoint になるよう取りたい。

$q \in \Delta^n$ をどう選んでも

$f^{-1}(q)$ は $\dim \le q-n$ の convex polyhedra の有限和 $\left(\because f \text{ は piecewise に } \mathbb{R}^q \to \mathbb{R}^n : \text{linear}\right)$

$\rightsquigarrow$ T は $\dim \le q-n+1$ の ―〃―

$\rightsquigarrow$ $f(T) \cap \Delta^m_\alpha$ はそれぞれ $\dim \le q-n+1$ の ―〃―　$(\because$ linear で次元は上がらない$)$

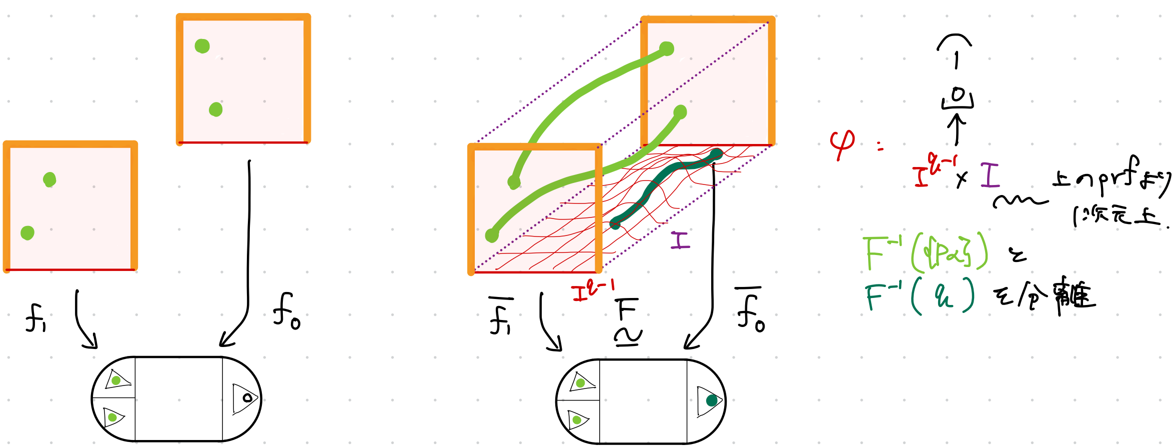
$\shortparallel$

$f(T \cap f^{-1}(\Delta^m_\alpha))$ f が PL な領域

$m \ge q-n+1$ なので $\{P_\alpha \in \Delta^m_\alpha \setminus f(T)\}$ が存在する！　Claim □

$q < m+n-2$ での単射性：

$$\pi_q(X-Q, X-Q-P) \longrightarrow \pi_q(X, X-P)$$



φ の下を渡す

① ここまで

② $\begin{array}{ccc} & B & \\ / & \smile & (\geq n)\text{-cell} \\ C & & \end{array}$ の場合

上の証明は$(\geq n)$-cellひとつの接着の場合にも回る。
I^qの像は有限個としか交わらないので①によりひとつずつ切除する。

③ $\begin{array}{ccc} & A & \\ (\geq m)\text{-cell} & \smile & \backslash \\ & & C \end{array}$ (i.e. 一般)の場合. $m+n-1$以上は考えなくてよい.

$$\begin{cases} A_k := (A \cap (\leq k)\text{-cell}) \cup C \subseteq A. \\ X_k := A_k \cup B \end{cases}$$

$$\begin{array}{ccc} (A_k, & A_{k-1}, & C) \\ \downarrow & \downarrow & \downarrow \\ (X_k, & X_{k-1}, & B) \end{array}$$ により

$$\begin{tikzcd}
\pi_{q+1}(A_k, A_{k-1}) \arrow[r] \arrow[d, "ア"] & \pi_q(A_{k-1}, C) \arrow[r] \arrow[d, "イ"] & \pi_q(A_k, C) \arrow[r] \arrow[d, "ウ"] & \pi_q(A_k, A_{k-1}) \arrow[r] \arrow[d, "エ"] & \pi_{q-1}(A_{k-1}, C) \arrow[d, "オ"] \\
\pi_{q+1}(X_k, X_{k-1}) \arrow[r] & \pi_q(X_{k-1}, B) \arrow[r] & \pi_q(X_k, B) \arrow[r] & \pi_q(X_k, X_{k-1}) \arrow[r] & \pi_{q-1}(X_{k-1}, B)
\end{tikzcd}$$

$(k = m+1)$ … ②で示した.

$(k > m+1)$ … ∘ $q < m+n-2$ において

- ∘ ア, エ は bij. ∵ セルひとつの接着なので②.
- ∘ イ, オ は bij. ∵ 帰納法の仮定

⇒ five lemma. より ウ : bij.

∘ $q = m+n-2$ において

- ∘ イ は surj. ∵ 帰
- ∘ エ は surj. ∵ ②
- ∘ オ は bij. ∵ 帰

⇒ five lemma. より ウ : surj.

Homotopy Excision

□